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$\mathcal{H}_\infty$ Control of Nonlinear Systems: A Convex Characterization

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Abstract

The so-called nonlinear $\mathcal{H}_\infty$-control problem in state space is considered with an emphasis on developing machinery with promising computational properties. Both state feedback and output feedback $\mathcal{H}_\infty$-control problems for a class of nonlinear systems are characterized in terms of continuous positive definite solutions of algebraic nonlinear matrix inequalities (NLMIs) which are convex feasibility problems.

1 Introduction

The simplicity of the characterization of state space $\mathcal{H}_\infty$-control theory together with its clear connections with traditional methods in optimal control [5] have stimulated several attempts to generalize the linear $\mathcal{H}_\infty$ results in state space to nonlinear systems [16, 8, 1, 11]. We will use the accepted but unfortunate misnomer "nonlinear $\mathcal{H}_\infty$" to describe this research direction, which will be pursued further in this paper, with an eye toward computational issues.

In those generalizations, a broadly accepted treatment in nonlinear $\mathcal{H}_\infty$-control theory is to assume that the (dynamic) output feedback $\mathcal{H}_\infty$-controllers have some separation structures. Under this assumption, some necessary or sufficient conditions for the $\mathcal{H}_\infty$-control problem to be (locally or globally) solvable are characterized in terms of some Hamilton-Jacobi equations or inequalities [16, 8, 1, 11, 17]. Whence, one of the major concerns in the state-space nonlinear $\mathcal{H}_\infty$-control theory is how to solve these Hamilton-Jacobi partial differential equations or inequalities, and progress along this line would be beneficial to applications of nonlinear $\mathcal{H}_\infty$-control theory. For example, van der Schaft [16] proposed an approach to approximate the local solutions of Hamilton-Jacobi equations.

In this paper, we propose an alternative approach to the state-space nonlinear $\mathcal{H}_\infty$-control problem, and characterize the solutions in terms of convex conditions instead of the Hamilton-Jacobi equations or inequalities. This is motivated by the fact that, essentially, the linear $\mathcal{H}_\infty$-control problem can be characterized as a convex problem which has very appealing computational property [2]. (The reader is referred to [14, 15, 10, 6, 9] for the treatments in linear case in terms of **linear matrix inequalities** (LMIs), which result in the convex problem.) There naturally arises a question: to what extent can convex characterizations be extended to deal with nonlinear systems? To this end, the convexity of the nonlinear $\mathcal{H}_\infty$-control problem will be examined, and the solvability conditions of a class of nonlinear $\mathcal{H}_\infty$-control problem are characterized in terms of some algebraic **nonlinear matrix inequalities** (NLMIs). Both state feedback and output feedback $\mathcal{H}_\infty$-control problems for a class of nonlinear systems are considered. In the output feedback case, the controllers are not required to have separation structures; the necessary conditions are characterized with three NLMIs. Therefore, a class of nonlinear $\mathcal{H}_\infty$-control problems can be solved via the convex optimization methods (some of the computation issues are considered in [12]). Unfortunately, unlike the linear case, the solution of the NLMIs by themselves are not sufficient to guarantee the existence of the required controllers, some additional condition is required, and the computational implications of the required additional constraints on the NLMI solutions are not clear at this moment.

This paper is organized as follows: In section 2, some background material related to the $\mathcal{L}_2$-gain analysis is provided. In section 3, the $\mathcal{H}_\infty$-control problem is stated. In section 4, the $\mathcal{H}_\infty$-control problems for both static and dynamic state-feedback are considered. In section 5,the output feedback $\mathcal{H}_\infty$-control problem is discussed; the necessary conditions for the solvability of this problem is characterized by three NLMIs.

The following conventions are made in this paper. $\mathbf{R}$ is the set of real numbers, $\mathbf{R}^+ := [0,\infty) \subset \mathbf{R}$. $\mathbf{R}^n$ is n-dimensional real Euclidean space; $\|\cdot\|$ stands for the **Euclidean norm**. For $\mathcal{B}_r$, it is understood to be the open ball in some Euclidean space with some radius $r > 0$ which is measured by Euclidean norm. $\mathbf{X}$ (or $\mathbf{X}_o$) is the state set which is a convex open subset of some Euclidean space and contains the origin. $\mathbf{R}^{n\times m}$ is the set of all $n \times m$ real matrices. The transpose of some matrix $M \in \mathbf{R}^{n\times n}$ is denoted by M^T. By $P > 0$ ($P \geq 0$) for some Hermitian matrix P we mean that the matrix is (semi-)positive definite. A function is said to be of class $\mathbf{C}^k$ if it is continuously differentiable k times; so $\mathbf{C}^0$ stands for the class of continuous functions.

2 $\mathcal{H}_\infty$-Performances and $\mathcal{H}_\infty$-Control Problem

In this section, some background material about $\mathcal{L}_2$-gain analysis of nonlinear systems is provided. The reader is referred [18, 16] for more details.

2.1 $\mathcal{L}_2$-Gain Analysis

Consider the following affine nonlinear time-invariant (NLTI) system:

$$G : \left\{ \begin{array}{l} \dot{x} = f(x) + g(x)w \\ z = h(x) + k(x)w \end{array} \right. \tag{2.1}$$

where $x \in \mathbf{R}^n$ is state vector, $w \in \mathbf{R}^p$ and $z \in \mathbf{R}^q$ are input and output vectors, respectively. It is assumed that $f, g, h, k \in \mathbf{C}^0$ are verctor or matrix valued function, and $f(0) = 0, h(0) = 0$. From now on we will assume the system evolves on a convex open subset $\mathbf{X} \subset \mathbf{R}^n$ containing the origin. Thus, $0 \in \mathbf{R}^n$ is the equilibrium of the system with $w = 0$.

Note that in many cases system (2.1) can be rewritten (nonuniquely) as the following form which is also used in this paper.

$$G : \left\{ \begin{array}{l} \dot{x} = A(x)x + B(x)w \\ z = C(x)x + D(x)w \end{array} \right. \tag{2.2}$$

where $x \in \mathbf{R}^n$ is state vector, $w \in \mathbf{R}^p$ and $z \in \mathbf{R}^q$ are input and output vectors, respectively. We will assume A, B, C, D are $\mathbf{C}^0$ matrix-valued functions of suitable dimensions.

Definition 2.1 *The system G (2.1) or (2.2) with initial state $x(0) = 0$ is said to have $\mathcal{L}_2$-gain* **less than or equal to** *γ for some $\gamma > 0$ if*

$$\int_0^T \|z(t)\|^2\, dt \leq \gamma^2 \int_0^T \|w(t)\|^2\, dt \tag{2.3}$$

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for all $T \geq 0$ and $w(t) \in \mathcal{L}_2[0, T]$.

The following results characterizes $\mathcal{L}_2$-gains for a class of nonlinear systems in terms of NLMIs.

Theorem 2.1 *Consider system (2.1)) with $R(x) = I - k^T(x)k(x) > 0$, it is asymptotically stable and has $\mathcal{L}_2$-gain ≤ 1 if there exist a $\mathbf{C}^1$ positive definite function $V : \mathbf{X} \rightarrow \mathbf{R}^+$ such that*

$$\begin{bmatrix} \frac{\partial V}{\partial x}(x)f(x) + h^T(x)h(x) & \frac{1}{2}\frac{\partial V}{\partial x}(x)g(x) + h^T(x)k(x) \\ \frac{1}{2}g^T(x)\frac{\partial V^T}{\partial x}(x) + k^T(x)h(x) & k^T(x)k(x) - I \end{bmatrix} \leq 0, \tag{2.4}$$

for all $x \in \mathbf{X}$.

Proof

From Schur's complement argument, it follows that (2.4) is equivalent to the following Hamilton-Jacobi inequality

$$\mathcal{H}(V, x) := \frac{\partial V}{\partial x}(x)f(x) + h^T(x)h(x) +$$

$$+(\frac{1}{2}\frac{\partial V}{\partial x}(x)g(x) + h^T(x)k(x))R^{-1}(x)(\frac{1}{2}g^T(x)\frac{\partial V^T}{\partial x}(x) + k^T(x)h(x)) \leq 0. \tag{2.5}$$

Thus,

$$\dot{V}(x) = \frac{\partial V}{\partial x}(x)(f(x) + g(x)w) \tag{2.6}$$

$$\leq \|w(t)\|^2 - \|z(t)\|^2 - \left\|R^{1/2}(x)(w(t) - w^*(t))\right\|^2$$

where $w^*(t) = R^{-1}(x)(k^T(x)h(x) - \frac{1}{2}g^T(x)\frac{\partial V^T}{\partial x}(x))$. The latter inequality follows by replacing $\frac{\partial V}{\partial x}(x)f(x)$ in (2.6) with the one solved by (2.5) and conducting *completion of squares*. Therefore,

$$\dot{V}(x) - (\|w(t)\|^2 + \|z(t)\|^2) \leq 0.$$

Take the integral from $t = 0$ to $t = T$, the above inequality implies that the system has $\mathcal{L}_2$-gain ≤ 1 since $V(x) \geq 0$.

□

Note that the asymptotic stability is guaranteed by the detectability assumption [16]. In fact, if $w = 0$, then $\dot{V}(x) \leq -\|z(t)\|^2$. Therefore $\dot{V}(x) = 0$ implies $z = h(x) = 0$, which further implies $x(t) \rightarrow 0$ as $t \rightarrow \infty$ if $[f(x), h(x)]$ is detectable, then the stability is confirmed by LaSalle theorem. In the following, we will not explicitly consider the stability issue.

Although (2.4) provides a convex characterization of the $\mathbf{C}^1$ positive definite function V which yields $\mathcal{L}_2$-gain ≤ 1, this fact has not been well exploited as in the linear case, where the corresponding conditions are also finite dimensional algebraic LMIs. It is possible to provide alternative characterizations which are more immediately comparable to the linear case, and which are useful in the synthesis problem, but at the price of increased conservatism. Consider system (2.2), suppose $V : \mathbf{X} \rightarrow \mathbf{R}^+$ satisfies (2.4). In addition, let $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ with some positive definite $\mathbf{C}^0$ matrix valued function $P : \mathbf{X} \rightarrow \mathbf{R}^{n \times n}$, then (2.4) becomes

$$\begin{bmatrix} x^T(A^T(x)P(x) + P(x)A(x) + C^T(x)C(x))x & x^T(P(x)B(x) + C^T(x)D(x)) \\ (B^T(x)P(x) + D^T(x)C(x))x & D^T(x)D(x) - I \end{bmatrix} \leq 0$$

It is clearly sufficient for the above NLMI to hold that

$$\begin{bmatrix} A^T(x)P(x) + P(x)A(x) + C^T(x)C(x) & P(x)B(x) + C^T(x)D(x) \\ B^T(x)P(x) + D^T(x)C(x) & D^T(x)D(x) - I \end{bmatrix} \leq 0, \tag{2.7}$$

for all $x \in \mathbf{X}$. (There indeed exists a gap between the above two characterizations, such an example is given in [13].) This observation is summarized as following theorem which gives alternative characterizations of the $\mathcal{L}_2$-gain of the system.

Theorem 2.2 *Consider the system G given by (2.2), suppose $I - D^T(x)D(x) > 0$. Given any $\mathbf{C}^0$ positive definite matrix-valued function $P : \mathbf{X} \rightarrow \mathbf{R}^{n \times n}$, the following statements are equivalent.*
(i) P satisfies NLMI (2.7).
(ii) P satisfies

$$\begin{bmatrix} A^T(x)P(x) + P^T(x)A(x) & P^T(x)B(x) & C^T(x) \\ B^T(x)P(x) & -I & D^T(x) \\ C(x) & D(x) & -I \end{bmatrix} \leq 0. \tag{2.8}$$

In addition, if there is a function $V : \mathbf{X} \rightarrow \mathbf{R}$ with $V(0) = 0$ such that $\frac{\partial V}{\partial x}(x) = 2xP^T(x)$, then the system has $\mathcal{L}_2$-gain ≤ 1.

Proof

The standard result about Schur complements yields the two inequalities are equivalent, since it is assumed $I - D^T(x)D(x) > 0$. As $V(x)$ is the positive definite function on $\mathbf{X}$ by lemma 5.1, the conclusion that the system has $\mathcal{L}_2$-gain ≤ 1 is confirmed by the preceding discussion. □

Note that the inequality conditions in Theorems 2.1 and 2.2 are affine in $V(x)$ ($P(x)$), and all such solutions form convex sets. These inequalities are actually state-dependent linear (or affine) matrix inequalities, but we will refer to them as **nonlinear matrix inequalities** (NLMIs) to emphasize their use in nonlinear problems. Some of the computation issues about solving such NLMIs are considered in [12].

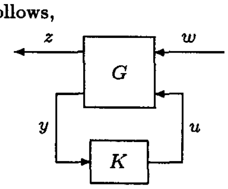
It should be emphasized that the existence of a $\mathbf{C}^0$ matrix-valued function $P : \mathbf{X} \rightarrow \mathbf{R}^{n \times n}$ which satisfies any of the NLMIs (2.7) and (2.8) is not enough to guarantee the system to have $\mathcal{L}_2$-gain ≤ 1; in this theorem, it is additionally required that there exists a function $V : \mathbf{X} \rightarrow \mathbf{R}$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P^T(x)$. (See lemma 5.1 for a characterization of a class of matrix-valued function $P : \mathbf{X} \rightarrow \mathbf{R}^{n \times n}$ which satisfies this additional requirement.)

We close this section by defining a stronger notion of $\mathcal{H}_\infty$-performance, which is in terms of theorem 2.2, and implies $\mathcal{L}_2$-gain ≤ 1. It is also possible to define a weaker notion which also implies $\mathcal{L}_2$-gain ≤ 1, using Theorem 2.1. Note that all of these versions are equivalent for linear systems, but for nonlinear systems, there is potentially a large gap.

Definition 2.2 *System (2.2) is said to have* **strong $\mathcal{H}_\infty$-performance** *if there is a $\mathbf{C}^0$ positive definite function $P(x) = P^T(x) > 0$ which satisfies any of inequalities (2.7) and (2.8) for all $x \in \mathbf{X}$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ for some $\mathbf{C}^1$ function $V : \mathbf{X} \rightarrow \mathbf{R}$.*

2.2 $\mathcal{H}_\infty$-Control Problems

The feedback configuration for the $\mathcal{H}_\infty$-control synthesis problem is depicted as follows,



where the nonlinear plant G has the following input-affine realization

$$G : \begin{cases} \dot{x} &= A(x)x + B_1(x)w + B_2(x)u \\ z &= C_1(x)x + D_{11}(x)w + D_{12}(x)u \\ y &= C_2(x)x + D_{21}(x)w + D_{22}(x)u \end{cases} \tag{2.9}$$

where $A, B_i, C_i, D_{ij} \in \mathbf{C}^0$; x, w, u, z, and y are assumed to have dimensions n, p_1, p_2, q_1, and q_2, respectively. The controller K to be designed also has an input-affine realization,

$$K : \begin{cases} \dot{\xi} &= \hat{A}(\xi)\xi + \hat{B}(\xi)y \\ u &= \hat{C}(\xi)\xi + \hat{D}(\xi)y \end{cases}$$

with $\hat{A}, \hat{B}, \hat{C}, \hat{D} \in \mathbf{C}^0$. It is assumed that the feedback system evolves in $(x, \xi) \in \mathbf{X} \times \mathbf{X}_o$, where $\mathbf{X}$ and $\mathbf{X}_o$ are open convex sets and contain the origins. The initial states for both plant and controller are $x(0) = 0$ and $\xi(0) = 0$.

The feedback closed loop system will be assumed to be **well-posed**. The following version of $\mathcal{H}_\infty$-control problem will be considered in this paper.

(Strong) $\mathcal{H}_\infty$-Control Problem: Find a feedback controller K (or a class controllers) if any, such that the closed-loop system has **strong $\mathcal{H}_\infty$-performance**. In this case, the feedback system has $\mathcal{L}_2$-gain ≤ 1.

The controllers to be sought in solving the above $\mathcal{H}_\infty$-Control Problem are called **strong $\mathcal{H}_\infty$-controllers**. Note that the stability issue is not explicitly touched here, as it is guaranteed by the observability assumption (see [8, 11]).

3 State Feedback $\mathcal{H}_\infty$-Control Problem

In this section, we consider the (strong) $\mathcal{H}_\infty$-control problem when the state x is directly measured and the controller is static feedback. It has been shown that the dynamic feedback controllers which have a separation structure can do no better than static feedback as far as the strong $\mathcal{H}_\infty$ control problem is concerned [12]. In this section, we consider the following system,

$$G_{SF}: \left\{ \begin{array}{rclclcl} \dot{x} & = & A(x)x & + & B_1(x)w & + & B_2(x)u \\ z & = & C_1(x)x & + & D_{11}(x)w & + & D_{12}(x)u \\ y & = & x & & & & \end{array} \right. \tag{3.1}$$

with $A, B_i, C_1, D_{ij} \in \mathbf{C}^0$. The state x, disturbance w, control input u, and regulated out put z have dimensions of n, p_1, p_2, and p_1, respectively; and $n + q_1 - p_2 \geq 0$. We assume that system evolves in $\mathbf{X}$, $rank \begin{bmatrix} B_2(x) \\ D_{12}(x) \end{bmatrix} = p_2$ and $D_{11}(x)D_{11}(x) < I$ for $x \in \mathbf{X}$.

Consider the system G_{SF}. Suppose the controller $u = F(x)x$ is such that the closed loop system

$$\left\{ \begin{array}{rclcl} \dot{x} & = & (A(x) + B_2(x)F(x))x & + & B_1(x)w \\ z & = & (C_1(x) + D_{12}(x)F(x))x & + & D_{11}(x)w \end{array} \right.$$

has strong $\mathcal{H}_\infty$-performance. By the definition 2.2, there is a $\mathbf{C}^0$ positive definite matrix-valued function $P = P^T : \mathbf{X} \to \mathbf{R}^{n \times n}$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ for some $\mathbf{C}^1$ function $V : \mathbf{X} \to \mathbf{R}^+$ and the following NLMI holds.

$$\begin{bmatrix} A_F^T(x)P(x) + P(x)A_F(x) & P(x)B_1(x) & C_F^T(x) \\ B_1^T(x)P(x) & -I & D_{11}^T(x) \\ C_F(x) & D_{11}(x) & -I \end{bmatrix} \leq 0.$$

where $A_F(x) = A(x) + B_2(x)F(x)$ and $C_F(x) = C_1(x) + D_{12}(x)F(x)$. Let $\mathcal{M}_{SF}(P, F, x)$ represent the left hand side of the above inequality. Define $T(x) := diag[P^{-1}(x), I, I]$, which is well-defined since $P(x) > 0$. Thus, $\mathcal{M}_{SF}(P, F, x) \leq 0$ if and only if $T^T(x)\mathcal{M}_{SF}(P, F, x)T(x) \leq 0$.. Let $X(x) = P^{-1}(x)$, which is of class $\mathbf{C}^0$, then

$$T^T(x)\mathcal{M}_{SF}(P, F, x)T(x) =$$
$$\tilde{\mathcal{M}}_{SF}(X, x) + \tilde{X}^T(x)F^T(x)\tilde{B}(x) + \tilde{B}^T(x)F(x)\tilde{X}(x) \leq 0 \tag{3.2}$$

where

$$\hat{\mathcal{M}}_{SF}(X, x) := \begin{bmatrix} X(x)A^T(x) + A(x)X(x) & B_1(x) & X(x)C_1^T(x) \\ B_1^T(x) & -I & D_{11}^T(x) \\ C_1(x)X(x) & D_{11}(x) & -I \end{bmatrix},$$

$$\tilde{X}(x) = \begin{bmatrix} X(x) & 0 & 0 \end{bmatrix}, \tilde{B}(x) = \begin{bmatrix} B_2^T(x) & 0 & D_{12}^T(x) \end{bmatrix}$$

By lemma 5.3, it follows that there is a solution $F(x)$ for (3.2) if and only if

$$X_\perp^T(x)\tilde{\mathcal{M}}_{SF}(X, x)X_\perp(x) \leq 0 \tag{3.3}$$

$$\tilde{B}_\perp^T(x)\tilde{\mathcal{M}}_{SF}(X, x)\tilde{B}_\perp(x) \leq 0 \tag{3.4}$$

for some $X_\perp(x)$ such that $span(X_\perp(x)) = \mathcal{N}(\tilde{X}(x))$, and $\tilde{B}_\perp(x)$ with $span(\tilde{B}_\perp(x)) = \mathcal{N}(\tilde{B}(x))$. Here $\mathcal{N}(\tilde{B}(x))$ for some matrix-valued function $\tilde{B}(x)$ stands for the distribution which annihilates all of the row vectors of $\tilde{B}(x)$.

Observe that (3.3) is guaranteed by $I - D_{11}^T(x)D_{11}(x) > 0$; (3.4) is actually written as

$$\tilde{B}_\perp^T(x) \begin{bmatrix} X(x)A^T(x) + A(x)X(x) & B_1(x) & X(x)C_1^T(x) \\ B_1^T(x) & -I & D_{11}^T(x) \\ C_1(x)X(x) & D_{11}(x) & -I \end{bmatrix} \tilde{B}_\perp(x) \leq 0 \tag{3.5}$$

Whence, we can conclude the following theorem.

Theorem 3.1 *The strong static state feedback $\mathcal{H}_\infty$-control problem is solvable if and only if there is a $\mathbf{C}^0$ matrix-valued function $X(x) = X^T(x) > 0$ with $\frac{\partial V}{\partial x}(x) = 2x^T X^{-1}(x)$ for some $\mathbf{C}^1$ function $V : \mathbf{X} \to \mathbf{R}^+$ such that for all $x \in \mathbf{X}$, the NLMI (3.5) holds.*

4 Output Feedback $\mathcal{H}_\infty$-Control Problem

In this section, we will consider the general strong $\mathcal{H}_\infty$-control problem; the system to be considered is

$$G: \left\{ \begin{array}{rclclcl} \dot{x} & = & A(x)x & + & B_1(x)w & + & B_2(x)u \\ z & = & C_1(x)x & + & D_{11}(x)w & + & D_{12}(x)u \\ y & = & C_2(x)x & + & D_{21}(x)w & + & D_{22}(x)u \end{array} \right. \tag{4.1}$$

where $A, B_i, C_i, D_{ij} \in \mathbf{C}^0$; x, w, u, z, and y are assumed to have dimensions n, p_1, p_2, q_1, and q_2, respectively; $n + p_1 \geq q_2$ and $n + q_1 \geq p_2$. Suppose the system (4.1) evolves in $\mathbf{X}$ which is a convex open subset of $\mathbf{R}^n$ and contains the origin; assume $rank \begin{bmatrix} B_2(x) \\ D_{12}(x) \end{bmatrix} = p_2$ and $rank \begin{bmatrix} C_1(x) & D_{21}(x) \end{bmatrix} = q_2$, and $D_{11}(x)D_{11}^T(x) < I$ for all $x \in \mathbf{X}$.

4.1 Necessary Conditions

Suppose the strong $\mathcal{H}_\infty$-controller is also of control-affine form:

$$K: \left\{ \begin{array}{rclcl} \dot{\xi} & = & \hat{A}(\xi)\xi & + & \hat{B}(\xi)y \\ u & = & \hat{C}(\xi)\xi & + & \hat{D}(\xi)y \end{array} \right.$$

with $\hat{A}, \hat{B}, \hat{C}, \hat{D} \in \mathbf{C}^0$. Suppose $\xi \in \mathbf{X}_o \subset \mathbf{R}^{n_d}$. The closed loop system evolves in $(x, \xi) \in \mathbf{X} \times \mathbf{X}_o$. We shall also assume that $I - \hat{D}(\xi)D_{22}(x)$ is invertible for all $(x, \xi) \in \mathbf{X} \times \mathbf{X}_o$ to assure the well-posedness of the feedback structure. Now take $x_c = \begin{bmatrix} x^T & \xi^T \end{bmatrix}^T$ to be the state of the closed loop system; define $R(x_c) := (I - \hat{D}(\xi)D_{22}(x))^{-1}$ for $(x, \xi) \in \mathbf{X} \times \mathbf{X}_o$. The feedback system has the following description:

$$\left\{ \begin{array}{rclcl} \dot{x_c} & = & A_c(x_c)x_c & + & B_c(x_c)w \\ z & = & C_c(x_c)x_c & + & D_c(x_c)w \end{array} \right.$$

for some matrix-valued functions $A_c(x_c), B_c(x_c), C_c(x_c)$, and $D_c(x_c)$ on $\mathbf{X} \times \mathbf{X}_o$. Define $\tilde{B}(x) := \begin{bmatrix} B_2^T(x) & 0 & D_{12}^T(x) \end{bmatrix}$ and $\tilde{C}(x) := \begin{bmatrix} C_2(x) & D_{21}(x) & 0 \end{bmatrix}$. The main theorem of this section is stated as follows.

Theorem 4.1 *Consider the (strong) output feedback $\mathcal{H}_\infty$-control problem with the plant defined as (4.1), let $B_\perp(x)$ is such that $\mathcal{N}(\tilde{B}(x)) = span(B_\perp(x))$, and $C_\perp(x)$ is such that $\mathcal{N}(\tilde{C}(x)) = span(C_\perp(x))$. Suppose there is a solution to the output feedback (strong) $\mathcal{H}_\infty$ control problem, then there are two $\mathbf{C}^0$ symmetrical matrix-valued functions $X, Y : \mathbf{X} \to \mathbf{R}^{n \times n}$, which are positive definite on $\mathbf{X}$, such that for all $x \in \mathbf{X} \subset \mathbf{R}^{n \times n}$ the following three NLMIs holds*

$$B_\perp^T(x) \begin{bmatrix} X(x)A^T(x) + A(x)X(x) & B_1(x) & X(x)C_1^T(x) \\ B_1^T(x) & -I & D_{11}^T(x) \\ C_1(x)X(x) & D_{11}(x) & -I \end{bmatrix} B_\perp(x) \leq 0 \tag{4.2}$$

$$C_\perp^T(x) \begin{bmatrix} A^T(x)Y(x) + Y(x)A(x) & Y(x)B_1(x) & C_1^T(x) \\ B_1^T(x)Y(x) & -I & D_{11}^T(x) \\ C_1(x) & D_{11}(x) & -I \end{bmatrix} C_\perp(x) \leq 0 \tag{4.3}$$

$$\begin{bmatrix} X(x) & I \\ I & Y(x) \end{bmatrix} \geq 0. \tag{4.4}$$

It is noted that all couples $(X(x), Y(x))$ satisfying the inequalities (i), (ii) and (iii) form a convex set. Therefore, theorem 4.1 provides a convex characterization to the necessary conditions for the strong output feedback $\mathcal{H}_\infty$-control problem to be solvable.

Proof

Define

$$A^a(x) := \begin{bmatrix} A(x) & 0 \\ 0 & 0 \end{bmatrix}, B_1^a(x) := \begin{bmatrix} B_1(x) \\ 0 \end{bmatrix},$$

$$B_2^a(x_c) := \begin{bmatrix} B_2(x) & 0 \\ \hat{B}(\xi)D_{22}(x) & I \end{bmatrix}, C_1^a(x) := \begin{bmatrix} C_1(x) & 0 \end{bmatrix},$$

$$D_{11}^a(x) := D_{11}(x), D_{12}^a(x) := \begin{bmatrix} D_{12}(x) & 0 \end{bmatrix},$$

$$C_2^a(x) := \begin{bmatrix} C_2(x) & 0 \\ 0 & I \end{bmatrix}, D_{21}^a(x) := \begin{bmatrix} D_{21}(x) \\ 0 \end{bmatrix}, D_{22}^a(x) := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and $F_c(x_c) := \begin{bmatrix} R(x_c)\hat{D}(\xi) & R(x_c)\hat{C}(\xi) \\ \hat{B}(\xi) & \hat{A}(\xi) \end{bmatrix}$. Thus

$$\begin{aligned} A_c(x_c) &= A^a(x) + B_2^a(x_c)F_c(x_c)C_2^a(x), \\ B_c(x_c) &= B_1^a(x) + B_2^a(x_c)F_c(x_c)D_{21}^a(x), \\ C_c(x_c) &= C_1^a + D_{12}^a F_c(x_c)C_2^a(x), \\ D_c(x_c) &= D_{11}^a(x) + D_{12}^a(x)F_c(x_c)D_{21}^a(x). \end{aligned}$$

Since the feedback system has strong $\mathcal{H}_\infty$-performance, by definition 2.2, there is a $\mathbf{C}^0$ positive definite matrix-valued function $P_c(x_c)$ on $\mathbf{X} \times \mathbf{X}_o$ such that

$$\begin{bmatrix} A_c^T(x_c)P_c(x_c) + P_c(x_c)A_c(x_c) & P_c(x_c)B_c(x_c) & C_c^T(x_c) \\ B_c^T(x_c)P_c(x_c) & -I & D_c^T(x_c) \\ C_c(x_c) & D_c(x_c) & -I \end{bmatrix} \leq 0. \tag{4.5}$$

Let $\mathcal{M}_c(P_c, x_c)$ represent the lesthand side of the above inequality, then

$$\mathcal{M}_c(P_c, x_c) = \mathcal{M}_a(P_c, x_c) + \tilde{C}^T(x_c)F_c^T(x_c)\tilde{B}(x_c)T_c(x_c) + \\ + T_c^T(x_c)\tilde{B}^T(x_c)F_c(x_c)\tilde{C}(x_c) \leq 0 \tag{4.6}$$

where $\mathcal{M}_a(P_c, x_c) =$

$$\begin{bmatrix} (A_a(x))^T P_c(x_c) + P_c(x_c)A^a(x) & P_c(x_c)B_1^a(x) & (C_1^a(x))^T \\ (B_1^a(x))^T P_c(x_c) & -I & (D_{11}^a(x))^T \\ C_1^a(x) & D_{11}^a(x) & -I \end{bmatrix},$$

$\tilde{B}(x_c) := \begin{bmatrix} (B_2^a(x_c))^T & 0 & (D_{12}^a(x))^T \end{bmatrix}, \tilde{C}(x_c) := \begin{bmatrix} C_2^a(x) & D_{21}^a(x) & 0 \end{bmatrix},$
and $T_c(x_c) := diag[P_c(x_c), I, I]$.

It follows from lemma 5.3 that (4.6) holds only if the following two inequalities hold (see lemma 5.3):

$$\tilde{B}_\perp^T(x_c)T_c^{-T}(x_c)\mathcal{M}_a(P_c, x_c)T_c^{-1}(x_c)\tilde{B}_\perp(x_c) \leq 0, \tag{4.7}$$

$$\tilde{C}_\perp^T(x_c)\mathcal{M}_a(P_c, x_c)\tilde{C}_\perp(x_c) \leq 0 \tag{4.8}$$

for all $\tilde{B}_\perp(x_c)$ with $span(\tilde{B}_\perp(x_c)) \in \mathcal{N}(\tilde{B}(x_c))$ and $\tilde{C}_\perp(x_c)$ with $span(\tilde{C}_\perp(x_c)) \in \mathcal{N}(\tilde{C}(x_c))$.

Next, we consider (4.7), notice that $\mathcal{N}(\tilde{B}(x_c)) = \mathcal{N}(\check{B}(x))$ for

$$\check{B}(x) := \begin{bmatrix} B_2^T(x) & 0 & 0 & D_{12}^T(x) \\ 0 & I & 0 & 0 \end{bmatrix}.$$

Thence, (4.7) holds if and only if

$$\check{B}_\perp^T(x)T_c^{-T}(x_c)\mathcal{M}_a(P_c, x_c)T_c^{-1}(x_c)\check{B}_\perp(x) \leq 0, \tag{4.9}$$

for all $\check{B}_\perp(x)$ with $span(\check{B}_\perp(x)) \in \mathcal{N}(\check{B}(x))$. On the other hand, notice that

$$T_c^{-T}(x_c)\mathcal{M}_a(P_c, x_c)T_c^{-1}(x_c) =$$

$$\begin{bmatrix} P_c^{-1}(x_c)(A^a(x))^T + A^a(x)P_c^{-1}(x_c) & B_1^a(x) & P_c^{-1}(x)(C_{11}^a(x))^T \\ (B_1^a(x))^T & -I & D_{11}^T(x) \\ C_1^a(x)P_c^{-1}(x_c) & D_{11}(x) & -I \end{bmatrix}$$

Since $P_c(x_c) = P_c(x, \xi)$ is invertible on $\mathbf{X} \times \mathbf{X}_o$, assume $X(x) = X^T(x) \in \mathbf{R}^{n \times n}$, which is positive definite and of class $\mathbf{C}^0$ on $\mathbf{X}$, is such that

$$P_c^{-1}(x, \phi(x)) = \begin{bmatrix} X(x) & X_1^T(x) \\ X_1(x) & X_0(x) \end{bmatrix}, \tag{4.10}$$

for some continuously differentiable function $\phi : x \mapsto \xi$ in $\mathbf{X}$ such that $\phi(\mathbf{X}) \subset \mathbf{X}_o$ (for example ϕ can be chosen as $\phi(x) = 0$). Therefore, (4.9), i.e. (4.7) implies

$$B_\perp^T(x) \begin{bmatrix} X(x)A^T(x) + A(x)X(x) & B_1(x) & X(x)C_1^T(x) \\ B_1(x) & -I & D_{11}^T(x) \\ C_1(x)X(x) & D_{11}(x) & -I \end{bmatrix} B_\perp(x) \leq 0$$

with $B_\perp : \mathbf{X} \to \mathbf{R}^{(n+q_1)\times(n+q_1-p_2)}$ such that $\mathcal{N}(B(x)) = span(\tilde{B}_\perp(x))$, where $\tilde{B}(x) := \begin{bmatrix} B_2^T(x) & 0 & D_{12}^T(x) \end{bmatrix}$.

Thus, the (4.2) is proved. Next, consider (4.8), if we take $Y(x) \in \mathbf{R}^{n \times n}$, which is of class $\mathbf{C}^0$, such that

$$P_c(x, \phi(x)) = \begin{bmatrix} Y(x) & Y_1^T(x) \\ Y_1(x) & Y_0(x) \end{bmatrix}, \tag{4.11}$$

Notice that $\tilde{C}(x_c)$ just depends on $x \in \mathbf{X}$, (4.8) implies (4.3)

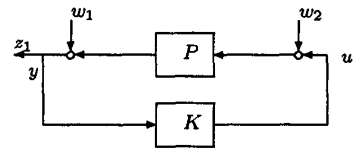
Finally, from lemma 5.2, it follows that (4.10) and (4.11) hold if and only if

$$\begin{bmatrix} X(x) & I \\ I & Y(x) \end{bmatrix} \geq 0.$$

Which is exactly (4.4. This concludes the proof. □

4.2 An Example

Consider the following system.



Where P is the nonlinear plant; K is the controller to be designed such that the output z_1 is regulated; y is the measured output; w_2 is a disturbance; and w_1 is sensor noise. The $\mathcal{H}_\infty$-control problem in this setting is formulated as: Give $\gamma > 0$, find K, if any, such that

$$\int_0^T (\|z_1\|^2 + \|u\|^2)dt \leq \gamma^2 \int_0^T (\|w_1\|^2 + \|w_2\|^2)dt, \forall T \in \mathbf{R}^+$$

In this example , the plant has the following realization:

$$P : \begin{cases} \dot{x} = e^x(w_2 + u) \\ z_1 = x + w_1 \\ y = x + w_1 \end{cases}$$

It is known that the optimal achievable $\mathcal{L}_2$-gain for this feedback system is $\gamma^* = \sqrt{2}$ [4], so let $w := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, $z := \frac{1}{\sqrt{2}} \begin{bmatrix} z_1 \\ u \end{bmatrix}$, and the scaled system is written in standard form as

$$\begin{cases} \dot{x} &= & & & \begin{bmatrix} 0 & e^x \end{bmatrix} w & + & e^x u \\ z &= & \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} x & + & \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} w & + & \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} u \\ y &= & x & + & \begin{bmatrix} 1 & 0 \end{bmatrix} w \end{cases}$$

The optimal controller $K = -1$ yields not only $\mathcal{L}_2$-gain ≤ 1 [4], but also, strong $\mathcal{H}_\infty$ performance, as is easily checked using theorem 2.2. Thus, the three NLMIs in theorem 4.1 should have solutions. We now verify this. We first consider NLMI in condition (i), which is as follows

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \sqrt{2}e^x \end{bmatrix} \begin{bmatrix} e^{2x} & \frac{1}{\sqrt{2}}X(x) & 0 \\ \frac{1}{\sqrt{2}}X(x) & -1/2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & \sqrt{2}e^x \end{bmatrix} \leq 0,$$

It turns out that all positive definite solutions satisfy $X(x) \leq e^x$.

The NLMI in condition (ii) is as follows

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & e^x Y(x) \\ 1/2 & -1/2 & 0 \\ e^x Y(x) & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \leq 0,$$

All positive definite solutions satisfy $Y(x) \leq e^{-x}$.

We then take two special solutions as

$$X(x) = e^x, \qquad Y(x) = e^{-x}.$$

Then $X(x)Y(x) = 1$, which implies condition (iii), i.e.,

$$\begin{bmatrix} e^x & 1 \\ 1 & e^{-x} \end{bmatrix} \geq 0.$$

4.3 Output Feedback and State Feedback

In this section, we further show that if the $\mathcal{H}_\infty$-control problem is solvable by output feedback, then it is also solvable by static state feedback if the output feedback controllers has separation structures(, and by static output injection [12]).

Suppose the output feedback strong $\mathcal{H}_\infty$-control problem for the given system (4.1) is solvable, then there is a $\mathbf{C}^0$ positive definite matrix-valued function $P_c(x_c)$ such that (4.5) holds. Moreover, there is a positive definite function $V_c(x_c)$ such that

$$\frac{\partial V_c}{\partial x_c}(x_c) = 2x_c^T P_c(x_c)$$

The following assumption is made.

Assumption 4.2 *There is a $\mathbf{C}^1$ function $\phi : x \mapsto \xi$ with $\phi(0) = 0$ such that $\frac{\partial V_c}{\partial \xi}(x, \xi)|_{\xi=\phi(x)} = 0$ with $(x, \xi) \in \mathbf{X} \times \mathbf{X}_o$.*

This assumption is not surprising. In fact, many dynamical controllers are observer-like-based [1, 8, 11]. In this case, the states x, ξ of a plant and its controller have a relation $\xi = \phi(x)$ for some $\mathbf{C}^1$ function $\phi : x \mapsto \xi$ with $\phi(0) = 0$ if the initial states satisfy $\xi(0) = \phi(x(0))$ and the disturbance is not imposed. The Lyapunov function for the closed loop system can be taken as $V_c(x,\xi) = V(x) + U(\xi - \phi(x))$ where V and U are Lyapunov functions of the state-feedback system and the observer. Thence, $\frac{\partial V_c}{\partial \xi}(x,\xi) = \frac{\partial U}{\partial e}(e)|_{e=\xi-\phi(x)}$. If e = 0, i.e. $\xi = \phi(x)$, then $\frac{\partial V_c}{\partial \xi}(x,\xi)|_{\xi=\phi(x)} = \frac{\partial U}{\partial e}(e)|_{e=0} = 0$. Therefore, V_c satisfies the assumption.

From the proof of the last theorem, it follows that (4.5) implies that there is $X(x) = X^T(x) \in \mathbf{R}^{n\times n}$, which is positive definite and of class $\mathbf{C}^0$ on $\mathbf{X}$, such that

$$P_c^{-1}(x,\phi(x)) = \begin{bmatrix} X(x) & X_1^T(x) \\ X_1(x) & X_0(x) \end{bmatrix},$$

for some continuously differentiable function $\phi : x \mapsto \xi$ on $\mathbf{X}$, and the NLMI (4.2) holds.

$\frac{\partial V_c}{\partial x_c}(x_c) = 2x_c^T X_c^{-1}(x_c)$ implies $\frac{\partial V_c}{\partial x_c}(x_c)X_c(x_c) = 2x_c^T$, or

$$\begin{bmatrix} \frac{\partial V_c}{\partial x}(x,\xi) & \frac{\partial V_c}{\partial \xi}(x,\xi) \end{bmatrix} X_c(x,\xi) = 2\begin{bmatrix} x^T & \xi^T \end{bmatrix}. \quad (4.12)$$

Take the function ϕ as in assumption 4.2, then (4.12) implies $\frac{\partial V_c}{\partial x}(x,\phi(x))X(x) = 2x^T$. Define $V(x) := V_c(x,\phi(x))$, then $V(x)$ is positive definite such that

$$\frac{\partial V}{\partial x}(x) = 2x^T X^{-1}(x). \quad (4.13)$$

Therefore, it can be concluded that the $\mathcal{H}_\infty$-control problem is indeed solvable in terms of static feedback. Thence, we have the following result.

Theorem 4.3 *If the strong $\mathcal{H}_\infty$-control problem is solvable in terms of the output feedback, then under assumption 4.2, it can also be solved in terms of static state feedback.*

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5 Appendix: Some Technical Results

The following result is quite standard, the reader is referred to, for instance, [12] for the proofs.

Lemma 5.1 *Suppose a vector-valued function $p : \mathbf{X} \to \mathbf{R}^n$ is of class $\mathbf{C}^1$; let $p(x) = [p_1(x), \cdots, p_n(x)]^T$ for $x \in \mathbf{X}$. Then there exists $V : \mathbf{X} \to \mathbf{R}$ such that*

$$\frac{\partial V}{\partial x}(x) = 2p^T(x)$$

if and only if

$$\frac{\partial p_i}{\partial x_j}(x) = \frac{\partial p_j}{\partial x_i}(x). \quad (5.1)$$

for all $x \in \mathbf{X}$ and $i,j = 1,2,\cdots,n$. Moreover, if (5.1) holds, then an function $V : \mathbf{X} \to \mathbf{R}$ with $V(0) = 0$ is given by

$$V(x) = 2x^T \int_0^1 p(tx)dt. \quad (5.2)$$

In addition, if $p(x) = P(x)x$ for some positive definite matrix-valued function such that $P(x)$, then $V(x)$ is also positive definite function.

The following result was first used in [15].

Lemma 5.2 *Let $X = X^T, Y = Y^T \in \mathbf{R}^{n\times n}$ be two positive definite matrices. Then there is a positive definite matrix $P = P^T \in \mathbf{R}^{(n+m)\times(n+m)}$ such that*

$$P = \begin{bmatrix} X & X_1^T \\ X_1 & X_0 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} Y & Y_1^T \\ Y_1 & Y_0 \end{bmatrix}$$

if and only if $\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0$.

Next, consider a matrix-valued function $B : \mathbf{M} \to \mathbf{R}^{m\times n}$, with $m \leq n$. Let $\Omega(B(x))$ be the **co-distribution** spanned by (smooth) co-vector fields of $B(x)$. It is assumed that each $x \in \mathbf{M}$ is a regular point of $\Omega(B(x))$, and the dimension of the co-distribution $\dim(\Omega(B(x))) = m$; thus, there is an $(n-m)$-dimensional (smooth) **distribution** $\mathcal{N}(B(x))$ which is the annihilator of $\Omega(B(x))$. The following result is standard, see [3, 2, 6, 9].

Lemma 5.3 *Consider the following matrix inequality*

$$Q(x) + U^T(x)F^T(x)V(x) + V^T(x)F(x)U(x) \leq 0 \quad (5.3)$$

with $Q = Q^T : \mathbf{M} \to \mathbf{R}^{m\times m}$, $U : \mathbf{M} \to \mathbf{R}^{r\times m}$ with $\dim\Omega(U(x)) = r < m$, and $V : \mathbf{M} \to \mathbf{R}^{s\times m}$ with $\dim\Omega(V(x)) = s < m$, then (5.3) has a solution $F : \mathbf{M} \to \mathbf{R}^{s\times r}$ if and only if

$$U_\perp^T(x)Q(x)U_\perp(x) \leq 0, \qquad V_\perp^T(x)Q(x)V_\perp(x) \leq 0$$

for some $U_\perp : \mathbf{M} \to \mathbf{R}^{m\times(m-r)}$ such that $span(U_\perp(x) = \mathcal{N}(U(x))$ and $V_\perp : \mathbf{M} \to \mathbf{R}^{m\times(m-s)}$ such that $span(V_\perp(x) = \mathcal{N}(V(x))$.

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